Large Elastic Deformation of an Inflatable Membrane of Revolution

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A general theory of large deformation of an anisotropic, elastic material has been applied to the problem of an inflatable, transversely isotropic membrane of revolution. Two states, the unstrained, and the strained states, are investigated. In the unstrained state, the thickness of the membrane is assumed to be uniform everywhere, but not in the strained state. In each state, the membrane is a smooth surface of revolution and is subjected to a uniform internal gas pressure. A set of static equilibrium equations is derived in terms of a strain energy density function and its derivatives, through the use of the calculus of variations. In order to carry out a numerical evaluation from the equilibrium equations, an explicit form of the energy density function has to be obtained. This function can be expressed in terms of two principal stretches. A finite-difference method is used to reduce the differential equations into the form of difference equations. The inflation of the membrane is determined. The results are compared with the isotropic case.

1. Introduction

IN recent years, the use of inflated structures has become attractive to aerospace engineers and to military equipment and communications satellite designers. Most of these structures are thin-walled shells filled with pressurized gas. The advantages of light weight and small folded volume make it possible for such structures to be carried conveniently to their destinations for performing their designed functions. In space exploration, such a structure can be ejected from a booster outside the atmosphere of the Earth to form a space station or a space laboratory. In re-entry operation, the space capsule as it descends down to the ocean surface can be supported by an inflated circular raft for a period of time while waiting to be picked up by a helicopter. Similar ideas are also applicable for pilots ejected from airplanes. An inflatable tent can be considered as another example of such structures. In many cases, such as the one being attempted in this investigation, a large elastic deformation theory must be used.

Many important researches in the area of large deformation of elastic membranes are summarized and recorded in the second edition of the book by Green and Adkin.[1] Most of the studies are based upon a well-known strain energy function which was suggested by Mooney.[2] Ericksen and Rivlin[3] extended the earlier studies from isotropic to transversely isotropic materials. They showed that the strain energy density function for such a material can be expressed in terms of five scalar invariants. The strain energy density function for anisotropic materials has been studied by Smith and Rivlin.[4] The authors discussed the restrictions imposed by symmetry on the form of this function, for elastic materials belonging to various crystal classes. The function was assumed to be a polynomial in g_{ij}, the deformation gradient tensor. This polynomial must be form-invariant under a group of transformations depending on the symmetry of materials. The problem of determination of the limitation imposed on the strain energy density function is reduced to the determination of a polynomial basis. Using the results obtained in Refs. 3 and 4, the strain energy density function is obtained in this study. The problem of an elastic membrane of revolution of isotropic materials has been formulated by Issacson.[5] Only the asymptotic solution of the equilibrium equations was discussed there. Recently, Wu[6] developed formal asymptotic series to represent the inflated membrane and found the series were power series in the parameter p^{-2} (p is the nondimensionalized pressure) and that the lowest-order term verified the formula obtained in Ref. 5. A part of the present analysis is an extension of Issacson's work to a transversely isotropic case.

Because of the complexity of the governing equations involved, the number of closed-form solutions to problems in the theory of finite deformations of elastic membranes obtained so far is limited. If a particular, simplified form is assumed for the strain energy density function, exact solutions may be obtained for certain problems. However, even with this assumption, the number of existing exact solutions still remains small. In the study on the large elastic deformations of membranes, approximate schemes are often used either in the form of finite element[7] or in other forms after certain clever transformations on the differential equations are made.[8,9] In this investigation, the equilibrium equations are solved numerically by a method suggested by Stoker.[10]

An investigation for an elastic membrane of revolution is carried out. Two equilibrium equations, which are expressed in terms of a general strain energy density function and its derivatives, are derived from the calculus of variations. These equations are valid for all materials. In order to find an explicit form of the equilibrium equations for this proposed problem, a particular form of the strain energy density function W is assumed. The strain energy density function is expressed as a polynomial in the arguments I_1, I_2, and K_1. An approximate form of the strain energy density function is suggested for the numerical calculations. By substituting this strain energy density function into the equilibrium equations derived, an explicit form of equilibrium equations is obtained. They are second-order, first-degree, coupled, nonlinear ordinary differential equations. These equations are solved through the use of a numerical method. The relations between deformations and loading parameters are

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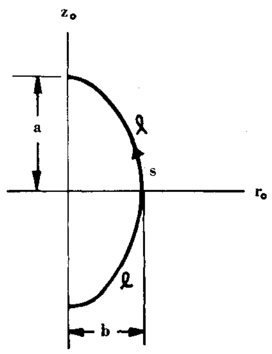


Fig. 1 Unstrained state.

obtained. A comparison between an isotropic and an anisotropic case is made.

2. Formulation of the Problem

2a. Unstrained State

The following assumptions are made: 1) the thickness of the membrane, h_o, in the unstrained state is small and uniform; 2) the weight of the membrane is so small that it can be ignored; 3) the membrane is incompressible, transversely isotropic; and 4) the membrane is a smooth surface of revolution generated by a line element as shown in Fig. 1. Due to rotational symmetry, only the first quadrant is studied. Let $z_o = z_o(s)$ and $r_o = r_o(s)$ (as shown in Fig. 1) be the axial and radial coordinates, respectively, in terms of the arc length s. The z_o-axis is the axis of revolution. As the internal pressure is further increased, large elastic deformations may take place in the membrane.

2b. Strained State

The membrane in the strained state is also assumed to be a smooth surface of revolution but with thickness h, and is highly inflated with a gas pressure q. Let $z = z(s)$ and $r = r(s)$ be the cylindrical polar coordinates (as shown in Fig. 2), where s is the arc length in the unstrained state, measured from the r_o-axis as shown in Fig. 1. It is assumed that there exists a strain energy function which is a single-valued, scalar function of a suitably defined strain tensor. Let W represent the strain energy density function which is measured per unit volume, and U represent the total strain energy, then

$$U = (2\pi)\int_{-l}^{l} W h_o r_o \, ds \qquad (1)$$

The variation of the strain energy is

$$\delta U = (2\pi)\int_{-l}^{l} h_o r_o \delta W \, ds \qquad (2)$$

Let V represent the total work done by a gas pressure q, therefore

$$V = q\left[\int \pi r^2 \, dz - \int \pi r_o^2 \, dz_o\right] = q\left[\int_{-l}^{l} \pi(r^2 z' - r_o^2 z_o') \, ds\right] \qquad (3)$$

where primes denote derivatives with respect to the arc length s. The variation of V is

$$\delta V = 2\pi q \int_{-l}^{l} (rz' \, \delta r - rr' \, \delta z) \, ds + \pi q r^2 \, \delta z \Big|_{-l}^{l}$$

Since r vanishes at $s = l$ and $s = -l$, the last term of the preceding equation is equal to zero. Therefore

$$\delta V = 2\pi q \int_{-l}^{l} (rz' \, \delta r - rr' \, \delta z) \, ds \qquad (4)$$

Since the variation of total energy must vanish, i.e., $\delta(U - V) = 0$ or

$$2\pi \int_{-l}^{l} [h_o r_o \delta W + qrr' \, \delta z - qrz' \, \delta r] \, ds = 0 \qquad (5)$$

If the strain energy density function can be expressed as a function of r, r', z and z', i.e., $W = W(r, r', z, z')$, Eq. (5) becomes

$$\int_{-l}^{l} \left[\left(\frac{\partial W}{\partial r}\delta r + \frac{\partial W}{\partial r'}\delta r' + \frac{\partial W}{\partial z}\delta z + \frac{\partial W}{\partial z'}\delta z'\right) h_o r_o + qrr' \, \delta z - qrz' \, \delta r\right] ds = 0 \qquad (6)$$

Integrating by parts, one obtains the following equilibrium equations:

$$\left.\begin{aligned} h_o r_o(\partial W/\partial r) - [h_o r_o(\partial W/\partial r')]' - qrz' &= 0 \\ h_o r_o(\partial W/\partial z) - [h_o r_o(\partial W/\partial z')]' + qrr' &= 0 \end{aligned}\right\} \qquad (7)$$

with the boundary conditions

$$\left.\begin{aligned} r &= 0, \\ \partial W/\partial z' &= 0‡ \end{aligned}\right\} \quad \text{at} \quad s = \pm l \qquad (8)$$

Equations (7) are the general equilibrium equations for all materials whose strain energy density function is a function of r, r', z and z'. The general results on the strain energy function are available in Refs. 1 and 11. It is known that the strain energy density function for an incompressible, transversely isotropic membrane is form invariant of I_1, I_2, and K_1 (Ref. 1), which can be expressed in terms of the two principal stretches in the following fashion

$$\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + 1/\lambda_1^2\lambda_2^2 \\ I_2 &= 1/\lambda_1^2 + 1/\lambda_2^2 + \lambda_1^2\lambda_2^2 \end{aligned} \qquad (9)$$

and

$$K_1 = \tfrac{1}{2}[(1/\lambda_1^2\lambda_2^2) - 1]$$

Since a cylindrical polar coordinate system (as shown in Fig. 1) is chosen for the derivation of the equilibrium equations, the principal stretches λ_1 and λ_2 of the membrane of revolution can be defined as

$$\lambda_1^2 = r'^2 + z'^2, \quad \lambda_2^2 = (r/r_o)^2 \qquad (10)$$

where primes again denote the derivatives with respect to the arc length s. Based upon the preceding results, it is shown in Ref. 15 that the strain energy density function W possesses the following properties:

$$\left.\begin{aligned} (\partial W/\partial z) &= 0 \\ z'(\partial W/\partial r') &= r'(\partial W/\partial z') \end{aligned}\right\} \qquad (11)$$

By using the preceding conditions, Eqs. (7) thus become

$$\left.\begin{aligned} h_o r_o(\partial W/\partial z') &= qr^2/2 \\ h_o r_o(\partial W/\partial r) - (qr^2 r'/2z')' &= qrz' \end{aligned}\right\} \qquad (12)$$

After making the substitution of a given W into Eq. (12), one will obtain the explicit form of the equilibrium equations.

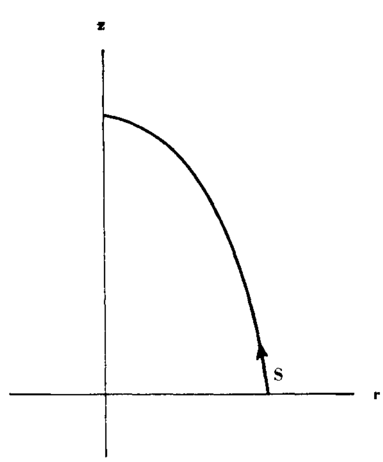


Fig. 2 Strained state.

‡ This condition leads to the symmetry condition, i.e., $z' = 0$ at the poles, after substituting the form of W adopted later in this paper.

2c. The Equilibrium Equations

It has been shown in Ref. 4 that under the conditions of smoothness available in the present problem, the strain energy density function can be approximated by a polynomial of its arguments to any desired degree of approximation. In the numerical calculations for this investigation, the strain energy density function is assumed to be

$$W = A_1(I_1-3)+A_2(I_2-3)+A_3K_1+A_4K_1^2+A_5K_1^2(I_1-3) \tag{13}$$

where A_1, A_2, A_3, A_4 and A_5 are material parameters, or

$$W = A_1[(I_1-3)+\Gamma_1(I_2-3)+\Gamma_2K_1+\Gamma_3K_1^2+\Gamma_4K_1^2(I_1-3)] \tag{14}$$

where

$$\Gamma_1 = A_2/A_1, \quad \Gamma_2 = A_3/A_1, \quad \Gamma_3 = A_4/A_1, \quad \text{and} \quad \Gamma_4 = A_5/A_1 \tag{15}$$

Substitution of Eqs. (9) and (10) into Eq. (14) gives an explicit form of the strain energy density function. In order to obtain an explicit form of the equilibrium equations, the partial derivatives $\partial W/\partial r$ and $\partial W/\partial z'$ are first obtained. By the chain rule, these partial derivatives can be written as

$$\partial W/\partial r = (\partial W/\partial I_1)(\partial I_1/\partial r)+(\partial W/\partial I_2)(\partial I_2/\partial r)+(\partial W/\partial K_1)(\partial K_1/\partial r) \tag{16}$$

$$\partial W/\partial z' = (\partial W/\partial I_1)(\partial I_1/\partial z')+(\partial W/\partial I_2)(\partial I_2/\partial z')+(\partial W/\partial K_1)(\partial K_1/\partial z') \tag{17}$$

After making some simplifications, a substitution of the appropriate derivatives into Eqs. (12) yields

$$(1-\eta)[1+\Gamma_1(r/r_o)^2+(\Gamma_4/4)(1-\eta)^2]+(\eta/2)[-\Gamma_2+(1-\eta)\times\{\Gamma_3+\Gamma_4[\![(r/r_o)^2+r'^2+z'^2+\eta-3]\!]\}] = qr^2/(4h_oA_1r_oz') \tag{18}$$

and

$$(1-\eta)[1+\Gamma_1(r'^2+z'^2)+(\Gamma_4/4)(1-\eta)^2]+\{\eta/[\![2(r/r_o)^2]\!]\}\times[-\Gamma_2+(1-\eta)\{\Gamma_3+\Gamma_4[\![(r/r_o)^2+r'^2+z'^2+z'^2+\eta-3]\!]\}]-(r^2r'/z')^2q/[4h_oA_1(r/r_o)] = qz'r_o/(2h_oA_1) \tag{19}$$

where η is defined as $1/[(r/r_o)^2(r'^2+z'^2)^2]$. Equations (18) and (19) are the static equilibrium equations where r and z' appear in the denominator of certain terms. This requires that neither r nor z' be equal to zero; but r and z' are zero at the poles, i.e., at $s = \pm l$ due to symmetry. Therefore the range of validity for the equilibrium equations must exclude the poles. The boundary conditions for the first quadrant can be stated as

$$\left.\begin{aligned} z(o) = r'(o) = 0 \\ \text{and} \quad r(l-) = z'(l-) = 0\end{aligned}\right\} \tag{20}$$

where

$$r(l-) = \lim_{s\to l-} r(s) \quad \text{and} \quad z'(l-) = \lim_{s\to l-} z'(s) \tag{21}$$

3. Solution of the Equilibrium Equations

In order to obtain the elastic deformations of the membrane, an attempt must be made to solve the equilibrium equations. Based upon the authors' knowledge, no analytic methods are available from the theory of differential equations for solving such a set of nonlinear, coupled, ordinary differential equations. The best approach appears to be a method from numerical analysis. An available approach is to solve a set of algebraic equations rather than a set of nonlinear differential equations. For this reason, a finite-difference method is used to reduce differential equations to difference equations. Once the algebraic equations have been obtained, a numerical solution can be obtained. If the values of variables r and z are known at a particular point, it is possible to calculate the derivatives of r and z at the next point from the preceding algebraic equations. Although the values of derivatives are known, one can not integrate directly to obtain the values of r and z at the next point. However by using Taylor's expansion, r can be expressed as

$$[r]_{s+\Delta s} = [r]_s+\Delta s[r']_s+\frac{(\Delta s)^2}{2!}[r'']_s+\frac{(\Delta s)^3}{3!}[r''']_s+\cdots$$

If the arc length Δs between two neighboring points is chosen sufficiently small, the expansion can be reduced approximately as

$$[r]_{s+\Delta s} = [r]_s+\Delta s[r']_s$$

Since the quantities on the right-hand side of the preceding expressions are known, $[r]_{s+\Delta s}$ can be calculated. By the same procedure, $[z]_{s+\Delta s}$ can be obtained. The outline given previously indicates that a finite-difference method can be used successfully to determine r and z for the next point. For calculational convenience, one lets

$$\left.\begin{aligned} r/r_o \equiv v, \quad r'/z' \equiv u, \quad qr_o/(4h_oA_1) \equiv \alpha, \quad 1+u^2 \equiv p \\ 1/(v^2pz'^2) \equiv \eta_1 \end{aligned}\right\} \tag{22}$$

where u, v, α, and p are nondimensional quantities. Then Eqs. (18) and (19) are rewritten as follows:

$$C_8z'^8-C_7z'^7-C_4z'^4-C_2z'^2-C_o = 0 \tag{23}$$

and

$$\alpha vu'r_o+2\alpha pz'-(1-\eta_1/v^2)[1+\Gamma_1pz'^2+\tfrac{1}{4}\Gamma_4(1-\eta_1)]-(\eta_1/2v^2)\{-\Gamma_2+(1-\eta_1)[\Gamma_3+\Gamma_4(v^2+pz'^2+\eta_1-3)]\} = 0 \tag{24}$$

Note that the coefficients C_i in Eq. (23) are given by

$$\begin{aligned} &C_o = 3\Gamma_4/4v^6, \quad C_2 = [(\Gamma_3-5\Gamma_4)/v^4+\Gamma_4/v^2](p/2) \\ &C_4 = [\Gamma_1+(1+\Gamma_2/2-\Gamma_3/2+7\Gamma_4/4)/v^2+\Gamma_4/4v^4-\Gamma_4/2]p^2 \\ &C_7 = \alpha v^2r_op^4, \quad C_8 = (1+v^2\Gamma_1+\Gamma_4/4)p^4 \end{aligned} \tag{25}$$

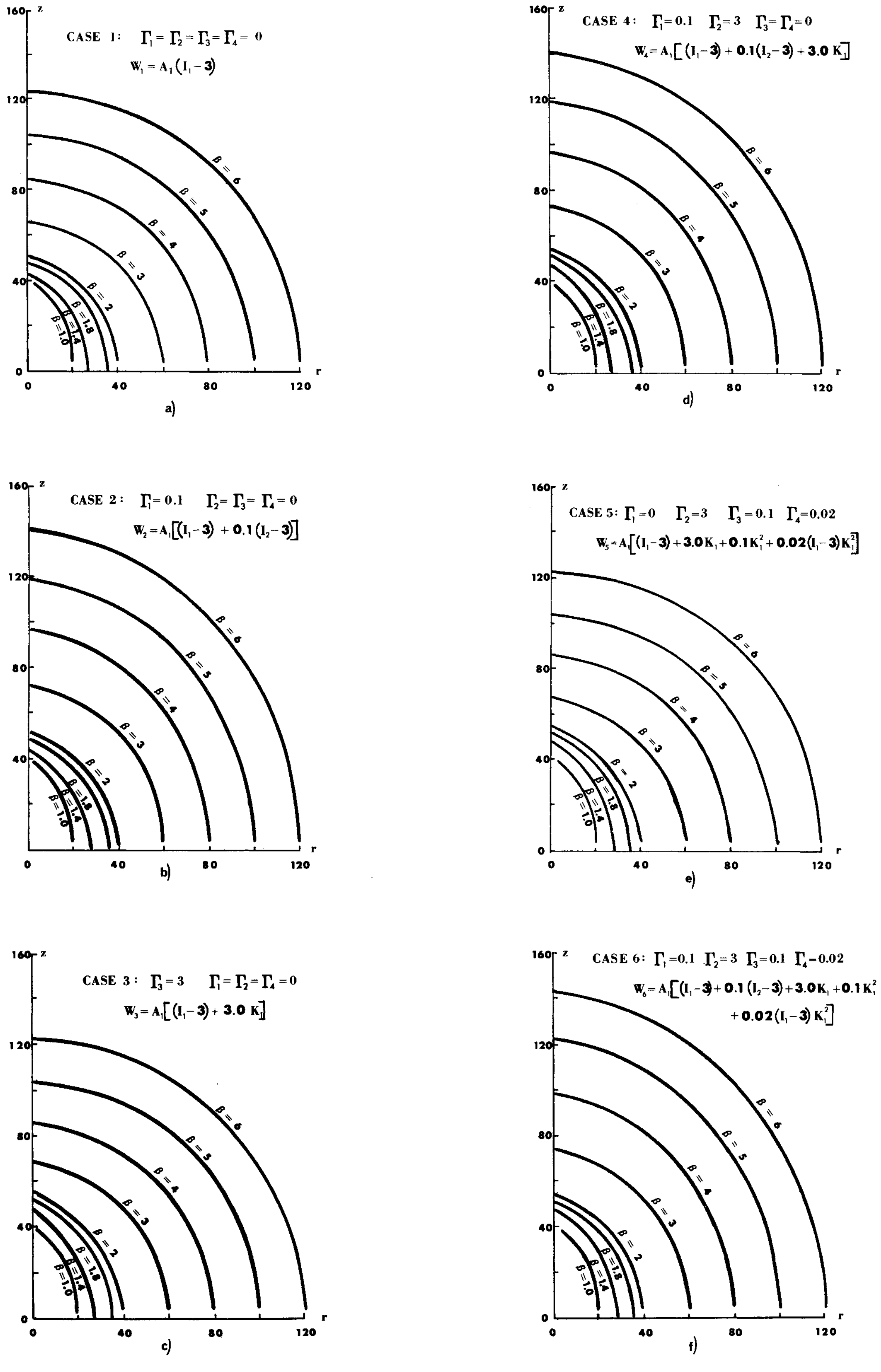
A finite-difference method is used to reduce the differential equation (24) to the difference equation

$$\alpha vr_o\frac{u|_{s+\Delta s}-u|_s}{\Delta s}+[\![2\alpha z'p-(1-\eta_1/v^2)[1+\Gamma_1pz'^2+\tfrac{1}{4}(1-\eta_1)]-\tfrac{1}{2}(\eta_1/2v^2)\{-\Gamma_2+(1-\eta_1)[\Gamma_3+\Gamma_4(v^4+pz'^2+\eta_1-3)]\}]\!] = 0 \tag{26}$$

In order to solve Eqs. (23) and (26) numerically with the aid of boundary conditions (20), it is better for computational purposes to treat the boundary value problem as an initial value problem.[10] This is done by adding another condition at $s = 0$, i.e., by prescribing the value of r at $s = 0$. The unstrained curve is assumed to be an ellipse characterized by $r_o^2/a^2+z_o^2/b^2 = 1$. The parameters a and b can be chosen arbitrarily. Once the values of a and b are chosen, the arc length l can be determined. Therefore the unstrained curve is given completely. In the calculations for this investigation, the values of a and b are assumed to be 40 and 20, respectively. Then the arc length l is approximately equal to 50. The unstrained curves ($\beta = 1.0$) are plotted in Figs. 3a–3f. The arc length is divided into 50 equal parts with 51 net points. Since r' is equal to zero at $s = 0$, in expression (22) $p = 1$. If the parameters Γ_1, Γ_2, Γ_3, and Γ_4 are prescribed, and r and z are given at $s = 0$, the values of C_i in Eqs. (25) are known at this point for a chosen value of α. Consequently, Eq. (23) becomes an eighth degree algebraic equation in z'. In order to be sure that there is only one value of z' for every value of r at the strained state, certain limitations must be imposed on the Γ_i. For instance, certain choices of the Γ_i, as given by the following expressions, make the C_i real and positive:

$$\begin{aligned} &C_o = 3\Gamma_4/4v^6 > 0, \quad \text{for} \quad \Gamma_4 > 0 \\ &C_2 = [(\Gamma_3-5\Gamma_4)/v^4+\Gamma_4/v^2](p/2) > 0, \quad \text{for} \quad 0 < \Gamma_4 < \Gamma_3/4 \\ &C_4 = [\Gamma_1-\Gamma_4/2+(1+\Gamma_2/2-\Gamma_3/2+7\Gamma_4/4)v^2+\Gamma_4/4v^6]p^2 > 0 \\ &\qquad \text{for} \quad \Gamma_3/2 < 1+\Gamma_1+\Gamma_2/2+3\Gamma_4/2 \quad \text{and} \quad \Gamma_4/2 < \Gamma_1 \\ &C_7 > 0 \\ &C_8 = (1+v^2\Gamma_1+\Gamma_4/4)p^4 > 0, \quad \text{for} \quad \Gamma_1 > 0, \quad \Gamma_4 > 0 \end{aligned}$$

Therefore, there is only one variation of signs in Eq. (23). By Descartes' rule of signs there is exactly one postive real root for z' in Eq. (23). However in Fig. 1, one recognizes that z increases as s increases; therefore the first derivative of z, with respect to s,



Figs. 3a–3f Inflation of the membrane.

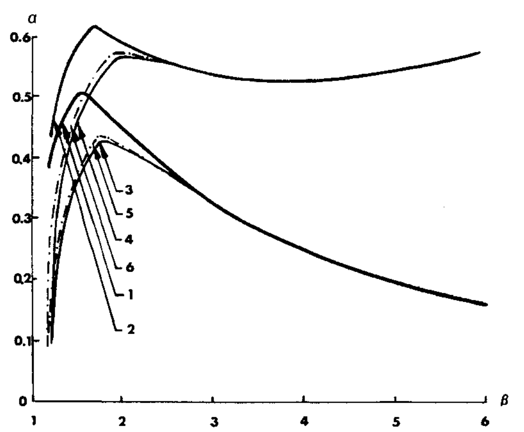


Fig. 4 Nondimensional pressure α vs nondimensional parameter β.

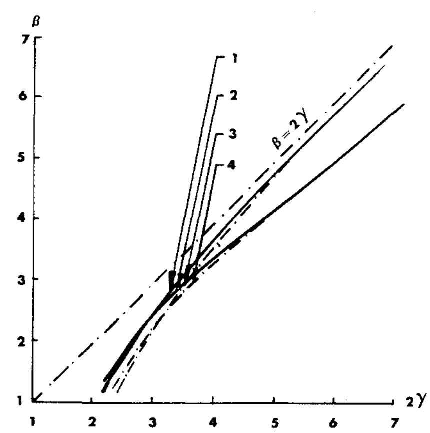


Fig. 6 Nondimensional parameters, β vs 2γ.

must be positive and real. Since there is exactly one positive, real root for z', z has only one value for every value of r. This positive root is calculated by iteration, which is an approximate value of z' for the first net point; therefore the values of r, z, r', and z' for the first net point are known. By employing the finite-difference method, the value of z for the second net point is calculated. Since r' is zero at the first net point, the value of r for the second net point is assumed to be the same as the value of r for the first net point. By substituting the values of r, r', z, and z' for the first net point into Eqs. (22) and (26), the value of r'/z' for the second net point is determined. Now, through the use of the values of r, z, r'/z' for the second net point, Eq. (23) is solved again for z'. This is an approximate value of z' for the second net point. Multiplication of r'/z' by z' gives the value of r' for the second net point. Employing the same method used for finding z at the second net point, r and z at the third net point can be obtained. The same process is repeated until arriving at the final net point. Thus the values of r, z, r', and z' for all net points are obtained. If the value of r for final net point is very close to zero, the chosen value of α is a good approximation for the loading parameter. Otherwise another value of α would have to be chosen. Repeating the same procedure, values of r, z, r', and z' can be obtained, until this condition is eventually satisfied.

First, the value of Γ_1, Γ_2, Γ_3, and Γ_4 are chosen to be zero. This corresponds to the Neo-Hookean material whose strain energy density function is given by

$$W_1 = A_1(I_1 - 3) \tag{27}$$

The corresponding deformations r and z are calculated and shown in Fig. 3a. Second, the parameters Γ_i are assumed to be as follows

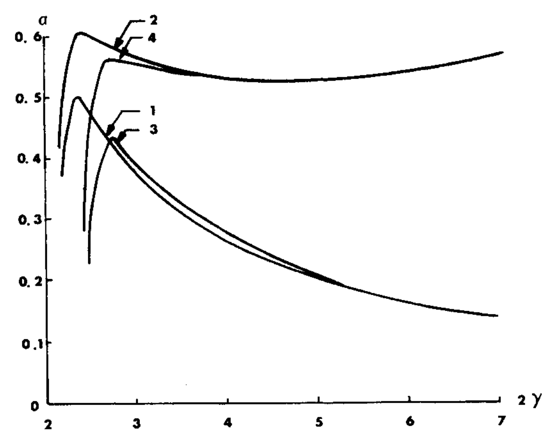


Fig. 5 Nondimensional pressure α vs nondimensional parameter 2γ.

$$\Gamma_1 = 0.1, \quad \Gamma_2 = \Gamma_3 = \Gamma_4 = 0 \tag{28}$$

This corresponds to Mooney's material whose strain energy density function is given by

$$W_2 = A_1[(I_1 - 3) + 0.1(I_2 - 3)] \tag{29}$$

The deformations r and z are calculated and shown in Fig. 3b. Both materials are isotropic. In addition, four more sets of material parameters are chosen. The elastic deformations of a membrane of revolution made from these materials are calculated and shown in Figs. 3c–3f. The values of Γ_i and the corresponding strain energy density functions are also listed there. The results shown in Fig. 4 indicate that the difference in deformation between case 3 and case 5 is rather insignificant. A similar conclusion can be made for case 4 and case 6.

Because the membrane studied in this investigation is neither spherical nor circular-cylindrical, both r/r_o and z/z_o vary from point to point along a meridian. Figures 4§ and 5 are given to indicate that α varies with the nondimensional parameters $\beta \equiv (r/r_o)_{s=o}$ and $\gamma \equiv (z/z_o)_{s=l}$, respectively. Figure 6 indicates how β varies vs 2γ.

From Figs. 3a–3f, one can see that the deformation along the meridian and circumferential directions are very large for a very large value of β. Therefore the corresponding thickness at the deformed state must be much smaller than that at the undeformed state, if the material is assumed to be incompressible. One may be interested in the determination of the deformed thickness. This can be done by using the condition of incompressibility, i.e., $\lambda_1 \lambda_2 \lambda_3 = 1$, where λ_3 is the principal stretch along the thickness.

4. Discussions and Conclusions

A study has been carried out on large elastic deformation of an inflatable, transversely isotropic membrane of revolution. Solutions for a membrane of revolution, based upon the various assumed values of Γ_i have been obtained in the present analysis. Accuracy of the numerical results on deformations of this problem depends on the order of K_1 involved in the strain energy density function. In the computational work of this analysis, the order of K_1 was chosen only up to two. More accurate results could be obtained by extending the strain energy density function up to the third or higher order of K_1. The numerical results on deformations, as shown in Figs. 3a–3f, indicate the deformation characteristics of a balloon-typed membrane of revolution, which was assumed initially to be an ellipse with $a = 40$ and $b = 20$. One recognizes that the increment of inflation along the r-axis is faster than that along the z-axis. Note that this can be thought of as a confirmation of the results of

§ Note that curve numbers 1, 2, 3, 4, 5, and 6 (as shown in Figs. 4–6) correspond to the case numbers as given in Figs. 3a–3f.

Issacson[5] and Wu[6] who have proved that an isotropic membrane of revolution deforms asymptotically to a spherical membrane.

The values of r/r_o and z/z_o vary along a meridian. Therefore the graphs of α vs r/r_o and z/z_o are different for every point on a meridian. Only two of these graphs, one for $s = 0$, and the other for $s = l$, were given in Figs. 4 and 5. From these figures, it is observed that the deformation is very sensitive to the parameter Γ_1 for large values of β and γ. A fall in the inflation pressure follows the initial rise as the inflation proceeds. But the pressure rises again for a higher degree of inflation. This phenomenon for the inflation of a spherical rubber balloon made of isotropic material was pointed out in Ref. 1. One can also see that the shape of curves 1 and 2 for Mooney and Neo-Hookean materials, respectively, is similar to that of curves 1 and 2 given in Fig. 12 of Ref. 12. The reason is that although the ratio of $(z)_{s=l}/(r)_{s=o}$ equals two at the unstrained state, it decreases very rapidly as inflation proceeds and approaches one at a certain pressure level as shown in Figs. 3a–3f. Therefore, the shape of the membrane will look more like a spherical shape with a very high internal pressure. Consequently, one can easily see that for every point on the membrane, the strain in the horizontal-circumferential direction is greater than that in the meridian direction. In Fig. 4, curves 3, 4, 5, and 6 are graphs of α vs β where the membrane of revolution is transversely isotropic. The departure from curve 2 to curve 4, or from curve 2 to curve 6, is apparent for small values of β. On the other hand, for large values of β, the difference is rather insignificant. The same conclusion can be made for the other curves in the same diagram. In this investigation the values of β were chosen only up to 6.0; no calculations were carried out for greater values of β. However, no limitations have been imposed on the magnitude of β in the theory developed here. Larger values of β can be used in the calculation without involving too much difficulty.

It should be pointed out that there are no experiments, either for an anisotropic or for an isotropic membrane of revolution, reported in the literature except the case where an isotropic, spherical membrane was investigated. Some experiments for such a membrane have been reported, for example, one by Rivlin and Saunders[13] and one by Alexander.[14] Based upon the reason of existence of an asymptotic solution for a membrane of revolution as mentioned before, one may therefore compare the results obtained in this investigation with the experimental ones. Since the Mooney strain energy density function is a fairly good approximation for uniaxial extension, one can also compare these results with those given in Ref. 12. From the comparison, one can see that the results obtained here on deformations for isotropic membrane of revolution are similar to the well-known results given in Ref. 12 for an isotropic, spherical balloon.

Since there is only one variation of signs in Eq. (23) for a certain choice of the Γ_i, the positive real root obtained is the only one which exists for Eq. (23). If there were two or more variations of signs, Eq. (23) may have more than one positive real root. Should Eq. (23) have more than one positive real root, then z would have more than one value for every value of r.

Physically, a membrane must have thickness. The unstrained thickness of the membrane has been assumed to be constant and small. But no restrictions on the strained thickness h, such as uniform thickness everywhere, have been made in this investigation. In a general case, the thickness at the strained state is not easy to calculate. Fortunately, in this investigation h can be obtained by the condition of incompressibility. It is observed and suggested here that a problem could be formulated without the assumption of constant thickness at the unstrained state.

Inflated structures have many potential applications. Some of the ideas have been successfully realized while the others are being included in many exploratory designs, such as those mentioned in Sec. 1. The results obtained in this investigation may have immediate applications to those problems.

The time used on UNIVAC 1108 computer was less than 15 min for the entire problem. This could be thought as a quite efficient method as far as the economy of the computer time is concerned. However, there were certain mathematical disadvantages in this method, such as if Eq. (23) had more than one sign variations there would be more than one positive real root. Fortunately, this did not happen. From a physical point of view, one could argue that this case would be very unlikely to happen in reality because it would lead to having more than one value of z for every value of r.

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